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1974 J. Phys. A: Math. Nucl. Gen. 7 213

(http://iopscience.iop.org/0301-0015/7/2/007)

J. Phys. A: Math., Nucl. Gen., Vol. 7, No. 2, 1974. Printed in Great Britain. © 1974

On the uncertainty relation in the coherent spin-state representation

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Received 30 May 1973, in final form 16 July 1973

Abstract. We intend to compute the product $(\delta S_x)^2(\delta S_y)^2$ where $(\delta S_{x,y})^2 = \langle S^2_{x,y}\rangle - \langle S_{x,y}\rangle^2$; averaging is performed in the coherent spin-state representation given by Radcliffe. After applying the Holstein–Primakoff transformation $\hat{S}_- = (2S)^{1/2}\hat{a}^\dagger$, $\hat{S}_+ = (2S)^{1/2}\hat{a}$ and $\mu = \alpha/(2S)^{1/2}$, and putting $S \to \infty$ we proceed from Radcliffe space into the Glauber space. After this procedure the product $(\delta S_x)^2(\delta S_y)^2$ becomes $(\delta x)^2(\delta p)^2$.

Using the Jackiw equation we have shown that the function $|\mu = 0\rangle$ is the only one which minimizes the uncertainty product, for every S.

The Radcliffe space is defined by (Radcliffe 1971)

$$|\mu\rangle = (1+|\mu|^2)^{-S}\sum_{p=0}^{2S}\left(\frac{2S!}{p!(2S-p)!}\right)^{1/2}\mu^p|p\rangle \tag{1}$$

where $|p\rangle$ is the eigenfunction of $\hat{S}_z$:

$$\hat{S}_z|p\rangle = (S-p)|p\rangle \qquad 0 \leqslant p \leqslant 2S. \tag{2}$$

As we know

$$\hat{S}_x = \tfrac{1}{2}(\hat{S}_+ + \hat{S}_-), \qquad \hat{S}_y = \tfrac{1}{2}\mathrm{i}(\hat{S}_- - \hat{S}_+) \tag{3}$$

where $\hat{S}_-$ and $\hat{S}_+$ are the spin-number creation and annihilation operators respectively.

Making use of Radcliffe's formulae for the matrix elements $\langle\lambda|S_+|\mu\rangle$ and $\langle\lambda|S_-|\mu\rangle$ we can write (Radcliffe 1971):

$$\langle\lambda|\hat{S}_x|\mu\rangle = \frac{S(\mu+\lambda^*)}{1+\lambda^*\mu}\langle\lambda|\mu\rangle \tag{4a}$$

$$\langle\lambda|\hat{S}_y|\mu\rangle = \frac{\mathrm{i}S(\lambda^*-\mu)}{1+\lambda^*\mu}\langle\lambda|\mu\rangle \tag{4b}$$

$$\langle\lambda|\hat{S}_z|\mu\rangle = \frac{S(1-\lambda^*\mu)}{1+\lambda^*\mu}\langle\lambda|\mu\rangle. \tag{4c}$$

It is well known that

$$\langle\lambda|\hat{S}^2_{x,y}|\mu\rangle = \frac{2S+1}{\pi}\int\frac{\langle\lambda|S_{x,y}|\nu\rangle\langle\nu|S_{x,y}|\mu\rangle\,\mathrm{d}^2\nu}{(1+|\nu|^2)^2}. \tag{5}$$

From (4*a*) and (4*b*), by performing standard computations, we get

$$\langle\lambda|S^2_{x,y}|\mu\rangle = \frac{1}{4}\left(\pm\frac{2S(2S-1)(\lambda^*)^2}{(1+\lambda^*\mu)^2}\pm\frac{2S(2S-1)\mu^2}{(1+\lambda^*\mu)^2}+\frac{4S^2\lambda^*\mu}{1+\lambda^*\mu}-\frac{2S(2S-1)(\lambda^*\mu)^2}{(1+\lambda^*\mu)^2}+\frac{2S}{1+\lambda^*\mu}+\frac{2S(2S-1)\lambda^*\mu}{(1+\lambda^*\mu)^2}\right)\langle\lambda|\mu\rangle. \tag{6}$$

Putting $\lambda = \mu$ into formulae (4*a*), (4*b*) and (6) gives the relevant mean values. Since $(\delta S_{x,y})^2 = \langle\hat{S}^2_{x,y}\rangle - \langle\hat{S}_{x,y}\rangle^2$, therefore

$$(\delta S_x)^2 = \frac{1}{4}\left(\frac{2S(2S-1)(\mu^*)^2}{(1+|\mu|^2)^2}+\frac{2S(2S-1)\mu^2}{(1+|\mu|^2)^2}+\frac{4S^2|\mu|^2}{1+|\mu|^2}-\frac{2S(2S-1)|\mu|^4}{(1+|\mu|^2)^2}+\frac{2S}{1+|\mu|^2}+\frac{2S(2S-1)|\mu|^2}{(1+|\mu|^2)^2}\right)-\frac{4S^2(\mathrm{Re}\,\mu)^2}{(1+|\mu|^2)^2} \tag{7a}$$

$$(\delta S_y)^2 = \frac{1}{4}\left(-\frac{2S(2S-1)(\mu^*)^2}{(1+|\mu|^2)^2}-\frac{2S(2S-1)\mu^2}{(1+|\mu|^2)^2}+\frac{4S^2|\mu|^2}{1+|\mu|^2}-\frac{2S(2S-1)|\mu|^4}{(1+|\mu|^2)^2}+\frac{2S}{1+|\mu|^2}+\frac{2S(2S-1)|\mu|^2}{(1+|\mu|^2)^2}\right)-\frac{4S^2(\mathrm{Im}\,\mu)^2}{(1+|\mu|^2)^2}. \tag{7b}$$

Following Radcliffe, we assume that μ represents a stereographic projection of the spin on the plane tangent to the sphere in its north pole. Therefore, we must write

$$\mu = \tan(\tfrac{1}{2}\theta)\,\mathrm{e}^{\mathrm{i}\phi}. \tag{8}$$

We can write the previous formulae more elegantly by using the above form for μ, thus

$$(\delta S_x)^2 = \tfrac{1}{4}S(1-2S)\cos\theta+\tfrac{1}{4}S(1+2S)-\tfrac{1}{2}S(1+\cos^2\phi)\sin^2\theta \tag{9a}$$

$$(\delta S_y)^2 = \tfrac{1}{4}S(1-2S)\cos\theta+\tfrac{1}{4}S(1+2S)-\tfrac{1}{2}S\sin^2\phi\sin^2\theta. \tag{9b}$$

Now we apply the Holstein–Primakoff transformations

$$\hat{S}_- = (2S)^{1/2}\hat{a}^\dagger, \qquad \mu = \frac{\alpha}{(2S)^{1/2}}$$

and

$$\hat{S}_+ = (2S)^{1/2}\hat{a}, \qquad \mu = \tan(\tfrac{1}{2}\theta_S)\,\mathrm{e}^{\mathrm{i}\phi_S}. \tag{10}$$

If μ maps a finite value on α then

$$\theta_S = 0. \tag{11}$$

As $S \to \infty$ we see that

$$\frac{(\delta S_x)^2}{S}\frac{(\delta S_y)^2}{S} = \tfrac{1}{4}. \tag{12}$$

This can be interpreted as the uncertainty relation on the Glauber state $|\alpha\rangle$ (Glauber 1963), since from (10) and the expression for a, $a^\dagger$ in terms of p, x, we have

$$\frac{(\delta S_x)^2(\delta S_y)^2}{S^2} = (\delta p)^2(\delta x)^2. \tag{13}$$

To discuss the uncertainty problem more comprehensively we must consider the Jackiw equation (Jackiw 1968):

$$\left(\frac{(\hat{X}-\langle X\rangle)^2}{(\delta X)^2}+\frac{(\hat{Y}-\langle Y\rangle)^2}{(\delta Y)^2}-\frac{2\hat{A}}{\langle A\rangle}\right)|\psi\rangle = 0 \tag{14}$$

which selects so called 'critical states' eg a class of functions $|\psi\rangle$ for which the product $(\delta X)^2(\delta Y)^2$ is constant. The operators $\hat{X}$, $\hat{Y}$ and $\hat{A}$ are related by the commutation rule

$$[\hat{X}, \hat{Y}] = \mathrm{i}\hat{A}. \tag{15}$$

We attempt to prove that, in the Radcliffe space, there are such 'critical states' related to the uncertainty product $(\delta S_x)^2(\delta S_y)^2$. In other words, we will solve the following Jackiw equation:

$$\left(\frac{(\hat{S}_x-\langle S_x\rangle)^2}{(\delta S_x)^2}+\frac{(\hat{S}_y-\langle S_y\rangle)^2}{(\delta S_y)^2}-\frac{2\hat{S}_z}{\langle S_z\rangle}\right)|\mu\rangle = 0. \tag{16}$$

We multiply this equation on the right-hand side by an arbitrary bra vector $\langle\lambda|$. In this way we get the new equation

$$\frac{\langle\lambda|(\hat{S}_x-\langle S_x\rangle)^2|\mu\rangle}{(\delta S_x)^2}+\frac{\langle\lambda|(\hat{S}_y-\langle S_y\rangle)^2|\mu\rangle}{(\delta S_y)^2}-\frac{2\langle\lambda|\hat{S}_z|\mu\rangle}{\langle S_z\rangle} = 0. \tag{17}$$

Having all matrix elements (see formulae (4*a*), (4*b*) and (6)) we may write the last equation finally in the form:

$$\begin{aligned}\Bigg[&\frac{1}{4(\delta S_x)^2}\left(\frac{2S(2S-1)(\lambda^*)^2}{(1+\lambda^*\mu)^2}+\frac{2S(2S-1)\mu^2}{(1+\lambda^*\mu)^2}+\frac{4S^2\lambda^*\mu}{1+\lambda^*\mu}-\frac{2S(2S-1)(\lambda^*\mu)^2}{(1+\lambda^*\mu)^2}+\frac{2S}{1+\lambda^*\mu}\right.\\ &\left.+\frac{2S(2S-1)\lambda^*\mu}{(1+\lambda^*\mu)^2}-\frac{2S(\lambda^*+\mu)}{1+\lambda^*\mu}\langle S_x\rangle+\langle S_x\rangle^2\right)+\frac{1}{4(\delta S_y)^2}\left(-\frac{2S(2S-1)(\lambda^*)^2}{(1+\lambda^*\mu)^2}\right.\\ &-\frac{2S(2S-1)\mu^2}{(1+\lambda^*\mu)^2}+\frac{4S^2\lambda^*\mu}{1+\lambda^*\mu}-\frac{2S(2S-1)(\lambda^*\mu)^2}{(1+\lambda^*\mu)^2}+\frac{2S}{1+\lambda^*\mu}+\frac{2S(2S-1)}{(1+\lambda^*\mu)^2}\lambda^*\mu\\ &\left.-\frac{2\mathrm{i}S(\lambda^*-\mu)}{1+\lambda^*\mu}\langle S_y\rangle+\langle S_y\rangle^2\right)-\frac{2S}{\langle S_z\rangle}\left(\frac{1-\lambda^*\mu}{1+\lambda^*\mu}\right)\Bigg]\langle\lambda|\mu\rangle = 0.\end{aligned} \tag{18}$$

This equation is very easily written in the form:

$$a(\lambda^*)^2+b\lambda^*+c = 0 \tag{19}$$

where a, b, c are functions of (μ, μ^*). Since equation (19) is satisfied for any arbitrary λ^*, therefore a, b, c must simultaneously disappear; this is only satisfied when $\theta = 0$ or $\mu = 0$. It can be seen from the formulae (9*a*) and (9*b*) that the function $|0\rangle$ is also the one that minimizes the uncertainty product.

Since for $S \to \infty$ we have $\theta_S = 0$ therefore, this property is induced on the Glauber space when S tends to infinity.

References

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